ACTION SELECTORS AND THE FIXED POINT SET OF A HAMILTONIAN DIFFEOMORPHISM

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ABSTRACT. We study the fixed point set of a Hamiltonian diffeomorphism of a closed symplectic manifold in the case where the number of points in the action spectrum is less than or equal to the cuplength of the manifold and, say, the manifold is symplectically aspherical. With this assumption we show that, for some degree greater than zero, the cohomology of the fixed point set must be nontrivial. In other words, there is a non-trivial cycle's worth of fixed points. We also establish a similar result for the size of the fixed point set when the action selectors satisfy a certain condition for a Hamiltonian defined on a monotone manifold.

CONTENTS

1. INTRODUCTION AND MAIN RESULTS

In this paper we study the relationship between action selectors and the fixed point set for a Hamiltonian diffeomorphism defined on a closed monotone symplectic manifold. In particular, we are interested in understanding the size of the fixed point set for a time dependent Hamiltonian whose action selectors satisfy a specific condition. We use the Arnold Conjecture as a starting point for the statement of the main results of this paper. The *Arnold Conjecture* states that every Hamiltonian diffeomorphism ϕ_H of a compact symplectic manifold (M, ω) possesses at least as many fixed points as a function $f : M \to \mathbb{R}$ possesses critical points. The weaker

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form of this conjecture asserts that the number of fixed points for ϕ_H is bounded below by the cuplength of the manifold plus one, i.e. $\#Fix(\phi_H) \geq CL(M)+1$. The $\mathbb{F}$-*cuplength of* M, denoted $CL(M)$, of a topological space M is the maximal integer k such that there exist classes $\alpha_1, \cdots, \alpha_k$ in the cohomology ring $H^{*>0}(M;\mathbb{F})$ satisfying

$$\alpha_1 \cup \cdots \cup \alpha_k \neq 0.$$

While the Arnold Conjecture is still an open problem in the case when M is a general rational, weakly monotone manifold, it has been proven in the symplectically aspherical case ([Flo89], [Hof88]). Suppose for the moment that M is symplectically aspherical. As is well known, one can use the basic properties and results concerning action selectors to prove the Arnold Conjecture when the Hamiltonian diffeomorphism has isolated fixed points, see e.g. [GG09] and [Sch98]. This is accomplished by using the spectrality properties of action selectors, meaning $c^\alpha(H) \in \mathcal{S}(H)$ where $\alpha \in H^*(M)$, H is a Hamiltonian, $c^\alpha(H)$ denoting our action selector, and $\mathcal{S}(H)$ the action spectrum. Using this fact, one is able to establish the following bound on the size of $\mathcal{S}(H)$:

$$\#\mathcal{S}(H) \geq CL(M)+1,$$

which in turn implies $\#Fix(\phi_H) \geq CL(M)+1$. Now, when H instead satisfies the condition $\#\mathcal{S}(H) < CL(M)+1$, it necessarily implies that the fixed point set for ϕ_H can't be isolated. As a result, this presents us with the following question: "How large" is the set $Fix(\phi_H)$ when $\#\mathcal{S}(H) < CL(M)+1$?
This leads us to one of the main results of the paper.

Theorem 1.1. *Suppose that (M,ω) is a symplectic manifold, which is closed and symplectically aspherical and H is a time dependent Hamiltonian with the property $\#\mathcal{S}(H) < CL(M)+1$. Let F denote the set of fixed points for the Hamiltonian diffeomorphism ϕ_H. Then $H^j(F) \neq 0$ for some $1 \leq j \leq 2n$.*

Remark 1.2. Because we are assuming that M is symplectically aspherical, this allows us to keep track of the various orbits. Hence, when M is not symplectically aspherical, then since the cappings of our contractible loops are nontrivial this presents the problem of determining what the geometrically distinct orbits are for H.

Theorem 1.1 will actually become an almost immediate corollary once the following result has been shown.

Theorem 1.3. *Let (M,ω) be a closed, monotone symplectic manifold, H be a time dependent Hamiltonian that is one-periodic in time and define F to be the fixed point set for the Hamiltonian diffeomorphism ϕ_H. Also assume that there exists cohomology elements $\alpha \in HQ^*(M)$, with $\alpha \neq 0$ and $HQ^*(M)$ the quantum cohomology of M, $\beta \in H^k(M)$ with $k > 0$, and satisfying the condition $c^{\alpha*\beta}(H) = c^\alpha(H)$. Then $H^k(F) \neq 0$ for $k = deg(\beta)$.*

Remark 1.4. It is worth noting that Theorem 1.3 also holds in the *negative* monotone case as well.

We would like to point out the similarity of Theorem 1.3 to a result due to Viterbo. In [Vit97] he deals with the Morse theoretic analogue of action selectors known as *critical value selectors* defined by the equation

$$c^\alpha_{LS}(f) = inf\{a \in \mathbb{R} \,|\, \alpha \neq 0 \,\text{in}\, H^*(M^a)\},$$

where $M^a = \{x \in M \,|\, f(x) \leq a\}$ and $f\colon M \to \mathbb{R}$ is at least C^1. Viterbo looks at the connection between the critical points of the function f and the critical value selectors. He establishes that when M is a Hilbert manifold, f a C^1-function on M satisfying the Palais-Smale condition, and for any $\alpha, \beta \in H^*(M)$ with cup-product $\alpha \cup \beta \neq 0$ in $H^*(M^a)$, then $c_{LS}^{\alpha\cup\beta}(f) \leq c_{LS}^{\alpha}(f)$. When $c_{LS}^{\alpha\cup\beta}(f) = c_{LS}^{\alpha}(f)$, F_a is the set of critical points of f at level $a = c_{LS}^{\alpha}(f)$; then β is nonzero on $H^*(F_a)$. As a result, $dim(F_a) \geq deg(\beta)$ and hence F_a is uncountable when $deg(\beta) \neq 0$.

Based on this result by Viterbo it is suspected that Theorem 1.3 should hold for a more general element β in the cohomology ring $HQ_-^*(M) := H^{*<2n}(M) \otimes \Lambda$. In future work we hope to extend Theorem 1.3 for $\beta \in HQ_-^*(M)$ and possibly when the manifold is weakly monotone as well.

1.0.1. *Organization of the paper.* In Section 2 we discuss our notational conventions and relevant definitions. Within Section 2 we have included several subsections where we outline the various tools and basic results concerning them. These subsections are meant to highlight the important features that will be used to prove Theorem 1.1 and Theorem 1.3. We do however make sure to point out useful references in order to aid the reader who is concerned with understanding their finer details. Then, in Section 3, we provide the proofs to Theorem 1.1 and Theorem 1.3.

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2. Preliminaries

2.1. **Conventions and basic definitions.**

The objective for this section of the paper is to set notation, definitions, and tools such as, Filtered Floer homology, Filtered Floer cohomology, quantum cohomology, the basics of the Ljusternik-Schnirelman theory, and Alexander-Spanier cohomology.

2.1.1. *Symplectic manifolds.* Throughout the paper we will assume that (M, ω) is a closed symplectic manifold, i.e. M is compact and $\partial M = \emptyset$. The manifold M is *monotone* if $[\omega]\big|_{\pi_2(M)} = \lambda c_1(M)\big|_{\pi_2(M)}$ for some non-negative constant λ. A *negative monotone* manifold satisfies the same condition, but with $\lambda < 0$. The manifold M is *rational* if $\langle [\omega], \pi_2(M)\rangle = \lambda_0\mathbb{Z}$, where $\lambda_0 > 0$. When $\langle c_1(M), \pi_2(M)\rangle$ is a discrete subgroup of $\mathbb{R}$, then we call the positive generator N of this subgroup the *minimal Chern number*. When M has the property $[\omega]\big|_{\pi_2(M)} = 0 = c_1(M)\big|_{\pi_2(M)}$, then M is called *symplectically aspherical*.

In this paper we will be working with time dependent Hamiltonians H. More specifically, we are going to be dealing with Hamiltonians which are one-periodic in time, meaning $H\colon S^1 \times M \to \mathbb{R}$ with $S^1 = \mathbb{R}/\mathbb{Z}$ and $H_t(\cdot) = H(t, \cdot)$ for $t \in S^1$. Let X_H denote the time dependent vector field that H generates, where X_H satisfies $i_{X_H}\omega = -dH$. Let ϕ_H^t denote the time dependent flow for the vector field X_H. In this paper we are interested in studying the time-one map of ϕ_H^t. We call the map $\phi_H := \phi_H^1$ a *Hamiltonian diffeomorphism*.

Let K and H be time dependent Hamiltonians, then we define $(K\#H)_t := K_t + H_t \circ (\phi_K^t)^{-1}$. The flow for the time dependent vector field generated by the Hamiltonian $K\#H$ is the composition $\phi_K^t \circ \phi_H^t$. As an aside, the composition $K\#H$ may not necessarily be one-periodic in time. If, however, $H_0 = 0 = H_1$, then the composition is one-periodic. One is able to impose this condition on H by reparametrizing H as a function of time without changing its time-one map. This allows us to treat $K\#H$ as a one-periodic Hamiltonian.

2.2. **Filtered Floer homology and filtered Floer cohomology.**

2.2.1. *Capped periodic orbits and filtered Floer homology.* In this section we begin by introducing the basics of Floer homology. We plan on only presenting the basic elements of Floer homology. For a more in depth discussion and for more on the specific details we refer the reader to [MS04], [HZ11], [BH05].

We start by looking at the contractible loops $x\colon S^1 \to M$. Since x is contractible we can attach a disk along the boundary of the loop, which produces a new mapping $u\colon D^2 \to M$ with $u\big|_{S^1}(t) = x(t)$. We call the map u a *capping* of the loop x and use the notation $\bar{x}$ to represent the pair (x, u). Let u_1 and u_2 be two cappings for the loop x. The two cappings are equivalent if the integrals of ω and $c_1(M)$ over the sphere formed by the connected sum $u_1\#(-u_2)$ is equal to zero. In the symplectically aspherical case all cappings of a fixed loop x are equivalent. Let $\mathcal{P}(H)$ be the set of contractible one-periodic solutions to X_H and $\bar{\mathcal{P}}(H)$ be the set of contractible capped one-periodic solutions to X_H.

The cappings of these loops allows us to define the *action functional* $\mathcal{A}_H$ for a time dependent Hamiltonian H. For a capped loop $\bar{x} = (x, u)$ we define

$$\mathcal{A}_H(\bar{x}) = -\int_u \omega + \int_0^1 H_t(x(t))dt.$$

The critical points for the action functional are the equivalence classes of capped loops $\bar{x}$ which are one-periodic solutions to the equation $\dot{x}(t) = X_H(t, x(t))$. The set of critical values for the action functional is called the *action spectrum* of H and is denoted by $\mathcal{S}(H)$. The action spectrum is a set of measure zero. In addition, when the manifold M is rational, $\mathcal{S}(H)$ is a closed set and implies it is a nowhere dense set ([HZ11]).

Following the terminology used in [SZ92], we will call a capped one-periodic orbit $\bar{x}$ of H *non-degenerate* if the pushforward $d\phi_H\colon T_{x(0)}M \to T_{x(0)}M$ has no eigenvalues equal to one. When all of the one-periodic orbits of H are non-degenerate, then we say H is *non-degenerate*. Note that the condition of degeneracy does not depend on the capping of the loop $x(t)$.

Whenever we are working with a non-degenerate Hamiltonian H we will end up with a finite number of elements in the set $\bar{\mathcal{P}}(H)$. By fixing a field $\mathbb{F}$ (e.g. $\mathbb{Z}_2, \mathbb{Q}, \mathbb{C}$) we can use the Conley-Zehnder index, denoted μ_{CZ}, to impose a grading on the vector space that is generated by the elements in the set $\bar{\mathcal{P}}(H)$ over $\mathbb{F}$. Define $CF_k^{(-\infty,b)}(H)$, for $b \in (-\infty, \infty]$ and b not an element in the set $\mathcal{S}(H)$, to be the vector space of sums given by

$$\sum_{\bar{x}\in\bar{\mathcal{P}}(H)} a_{\bar{x}}\bar{x},$$

with $a_{\bar{x}} \in \mathbb{F}$, $\mu_{CZ}(\bar{x}) = k$, $\mathcal{A}_H(\bar{x}) < b$, and the number of terms in the sum with $a_{\bar{x}} \neq 0$ is finite. There is a linear boundary operator $\partial\colon CF_k^{(-\infty,b)}(H) \to$

$CF_{k-1}^{(-\infty,b)}(H)$, where for $\bar{x} \in \bar{\mathcal{P}}(H)$ with $\mu_{CZ}(\bar{x}) = k$ is defined to be

$$\partial \bar{x} = \sum_{\mu_{CZ}(\bar{y})=k-1} n(\bar{x}, \bar{y})\bar{y}$$

and $\partial^2 = 0$. When $\mathbb{F} = \mathbb{Z}_2$ the number $n(\bar{x}, \bar{y})$ counts the number of components in the 1-dimensional moduli space $\mathcal{M}(\bar{x}, \bar{y})$ *mod* 2. For a more general field $\mathbb{F}$, the number $n(\bar{x}, \bar{y})$ is a bit more involved to describe and we refer the reader to [FH93]. One can further define $CF_k^{(a,b)}(H) := CF_k^{(-\infty,b)}(H)/CF_k^{(-\infty,a)}(H)$, for $-\infty \leq a < b \leq \infty$ not in $\mathcal{S}(H)$. The above construction results in what is known as the *filtered Floer homology* of H and is denoted by $HF_*^{(a,b)}(H)$. Note when $(a, b) = (-\infty, \infty)$ we end up with the standard Floer homology $HF_*(H)$.

Since the results of this paper deal with Hamiltonians that are degenerate, it is worth pointing out that filtered Floer homology can be defined in the degenerate case. Take H to be a Hamiltonian on M with $a, b \notin \mathcal{S}(H)$ and M to be a rational manifold. By virtue of the fact that we can always find a non-degenerate Hamiltonian $\tilde{H}$ from an arbitrarily small perturbation of H it allows us to define

$$HF_*^{(a,b)}(H) = HF_*^{(a,b)}(\tilde{H}).$$

2.2.2. *Filtered Floer cohomology.* Now that the basics of Floer homology have been presented it then becomes a fairly straightforward process to explain the setup for the Floer cohomology.

We again take H to be a non-degenerate Hamiltonian and R to be a fixed commutative ring. Define the cochain complex $CF^*(H)$ to be the set of functions $\alpha : \bar{\mathcal{P}}(H) \to R$ that satisfy the finiteness condition $\#\{\bar{x} \in \bar{\mathcal{P}}(H) \,|\, \alpha(\bar{x}) \neq 0, \mathcal{A}_H(\bar{x}) \leq c\} < \infty$ for every real number c. Define the *filtered Floer chain complex* by $CF^k_{(-\infty,b)}(H)$ for $b \in (-\infty, \infty]$ and b not in $\mathcal{S}(H)$ to be the vector space of formal sums

$$\sum_{\bar{x} \in \bar{\mathcal{P}}(H)} \alpha_{\bar{x}} \alpha,$$

where $\mu_{CZ}(\bar{x}) = k$ and $\alpha(\bar{x}) \neq 0$. Also, using the same numbers $n(\bar{x}, \bar{y})$ from the Floer chain complex determines a linear coboundary operator $\delta : CF^k_{(-\infty,b)}(H) \to CF^{k+1}_{(-\infty,b)}(H)$, given by

$$\delta\alpha(\bar{x}) = \sum_{\mu_{CZ}(\bar{y})=k+1} n(\bar{x}, \bar{y})\alpha(\bar{y}),$$

where $\bar{x} \in \bar{\mathcal{P}}(H)$, $\mu_{CZ}(\bar{x}) = k$, and satisfies $\delta^2 = 0$. We define $CF^k_{(a,b)}(H) := CF^k_{(-\infty,b)}(H)/CF^k_{(-\infty,a)}(H)$, for $-\infty \leq a < b \leq \infty$ which are not elements of $\mathcal{S}(H)$. This results in giving us the *filtered Floer cohomology* of H and is denoted by $HF^*_{(a,b)}(H)$.

Just like the case of Floer homology, we can also define the filtered Floer cohomology for a degenerate Hamiltonian H by choosing a non-degenerate Hamiltonian $\tilde{H}$ that is sufficiently close to H and setting

$$H^*_{(a,b)}(H) = H^*_{(a,b)}(\tilde{H}).$$

2.3. **Quantum cohomology.** The *quantum cohomology* is obtained by tensoring the cohomology ring $H^*(M)$ with the Novikov ring Λ over a field $\mathbb{F}$, i.e.

$$HQ^*(M) = H^*(M) \otimes_{\mathbb{F}} \Lambda.$$

Let $\alpha \in H^*(M)$ and $A \in \Lambda$, then the degree of the generator $\alpha \otimes e^A$ is given by $deg(\alpha \otimes e^A) = deg(\alpha) + I_{c_1}(A)$. This leads to a grading on $HQ^*(M)$ where the cohomology classes of degree k are elements in the direct sum

$$QH^k(M) = \oplus_{i=0}^k H^i(M) \otimes_{\mathbb{F}} \Lambda^{k-i}.$$

One can also take $\Lambda = \mathbb{Z}[q, q^{-1}]$, which is the ring of Laurent polynomials.[1] This is done by sending the element $e^A \mapsto q^{c_1(A)/N}$, where N is the minimal Chern number, and q is a variable of degree $2N$. This results in the isomorphism $QH^k(M) \cong QH^{k+2N}(M)$ for all k when we multiply the quantum cohomology elements by q.

There is also a product structure defined on the quantum cohomology: let $\alpha \in HQ^k(M)$, $\beta \in HQ^l(M)$, then the *quantum cup product* of α with β is given by

$$\alpha * \beta = \sum_A (\alpha * \beta)_A q^{c_1(A)/N},$$

where $deg(\alpha * \beta) = deg(\alpha) + deg(\beta)$ and each of the cohomology classes $(\alpha * \beta)_A \in H^{k+l-2c_1(A)}(M)$ are defined by the Gromov-Witten invariants $GW^M_{A,3}$. The invariants $GW^M_{A,3}$ satisfy

$$\int_c (\alpha * \beta)_A = \int_M (\alpha * \beta)_A \cup \eta = GW^M_{A,3}(a, b, c),$$

where $c \in H_{k+l-2c_1(A)}(M)$, $a = PD(\alpha)$, $b = PD(\beta)$, $c = PD(\eta)$ and $deg(a) + deg(b) + deg(c) = 4n - 2c_1(A)$.[2] When this degree condition is not met, then $GW^M_{A,3}(a, b, c) = 0$. Also, when $c_1(A) = 0$ then $(\alpha * \beta)_A$ reduces to the cup product $\alpha \cup \beta$. One can also find a detailed presentation of Gromov-Witten invariants in [MS04].

2.4. **The classical Ljusternik–Schnirelman theory: critical value selectors and action selectors.** In order to prove Theorems 1.1 and 1.3 we will use tools from the Ljusternik-Schnirelmann theory known as critical value selectors and action selectors. The action selectors, also known as spectral invariants in the literature, are the Floer theoretic version of critical value selectors.

Definition 2.1 (Critical Value Selectors)**.**

Let M be a n-dimensional manifold and $f \in C^\infty(M)$. For any $u \in H_*(M)$ we define the *critical value selector* by the formula

$$\begin{aligned} c_u^{LS}(f) &= inf\{a \in \mathbb{R} \mid u \in im(i^a)\} \\ &= inf\{a \in \mathbb{R} \mid j^a(u) = 0\}, \end{aligned}$$

where $i^a \colon H_*(\{x \in M \mid f(x) \leq a\}) \to H_*(M)$ and $j^a \colon H_*(M) \to H_*(M, \{x \in M \mid f(x) \leq a\})$ are the natural "inclusion" and "quotient" maps respectively.

One can think of the critical value selectors geometrically in terms of minimax principles. Take a nonzero homology class $u \in H_*(M)$, then one can think of $c_u^{LS}(f)$

[1]Note that instead of $\mathbb{Z}$ one can replace it with any commutative ring R with unit.

[2]Here, and in throughout the rest of the paper, the notation "PD" stands for the Poincaré dual.

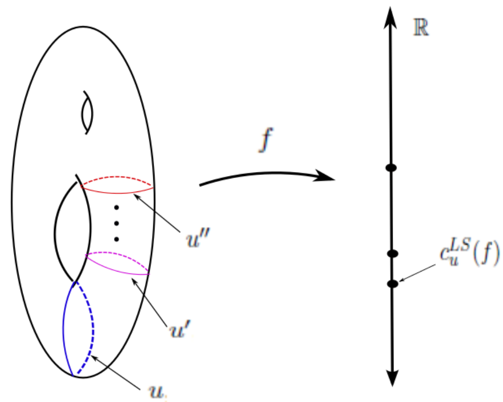

FIGURE 1. Critical value selector.

to be the maximum value f takes on the any representative cycle $u' \in u$ that has been "pushed down" as far as possible within the manifold M, see Figure 1. So, when f is a Morse function then we can write

$$c_u^{LS}(f) = min\ \ max_{[u']=u}\{f(x)\,|\,x \in u'\}.$$

The following is a listing of some useful properties concerning critical value selectors.

- By definition, $c_0^{LS}(f) = -\infty$. When $f \equiv$ const then $c_u^{LS}(f) \equiv$ const as well and, for any nonzero $\lambda \in \mathbb{F}$, $c_{\lambda u}^{LS}(f) = c_u^{LS}(f)$. For any function f we have
$$c_1^{LS}(f) = min(f) \leq c_u^{LS}(f) \leq max(f) = c_{[M]}^{LS}(f).$$
- Continuity: $c_u^{LS}(f)$ is Lipschitz with respect to the C^0-topology.
- Triangle Inequality: $c_{u\cap w}^{LS}(f+g) \leq c_u^{LS}(g) + c_w^{LS}(g)$.
- Criticality or minimax principle: $c_u^{LS}(f)$ is a critical value of f.
- $c_{u\cap w}^{LS}(f) \leq c_u^{LS}(f)$, also, if $w \neq [M]$ and the critical points of f are isolated, we have strict inequality $c_{u\cap w}^{LS}(f) < c_u^{LS}(f)$.

2.4.1. *The Hamiltonian Ljusternik–Schnirelman theory: action selectors.*

In this section we present the definition and outline the fundamental properties pertaining to action selectors on cohomology. The action selectors are defined in a somewhat similar manner, where one big difference is the function $f\colon M \to \mathbb{R}$ is replaced by the action functional $\mathcal{A}_H$ for some Hamiltonian H. There are numerous sources on the subject of spectral invariants. Some of the first instances concerning the theory can be found in [HZ11], [Vit92]. A thorough treatment of the symplectically aspherical case can be found in [Sch00]. Other known sources can be found in [HZ11], [Vit92], [EP03], [EP09], [Gin05], [GG09], [MS04]. In our paper we will be primarily following the definitions and results found in [Oh05].

Definition 2.2 (Action Selectors on Cohomology)**.** For any nonzero element $\alpha \in HQ^*(M) \cong HF^*(H)$ we define the *action selector on cohomology* by the formula

$$\begin{aligned} c^\alpha(H) &= inf\{a \in \mathbb{R} - \mathcal{S}(H)\,|\,PD(\alpha) \in im(i_*^a)\} \\ &= inf\{a \in \mathbb{R} - \mathcal{S}(H)\,|\,j_*^a(PD(\alpha)) = 0\}, \end{aligned}$$

where $i^a_*\colon HF^{(-\infty,a)}_*(H) \to HF_*(H)$ and $j^a_*\colon HF_*(H) \to HF^{(a,\infty)}_*(H)$ are the "inclusion" and "quotient" maps respectively.

When H is a non-degenerate Hamiltonian we can write

$$c^\alpha(H) = \inf_{[\sigma]=a} \mathcal{A}_H(\sigma),$$

where $a = PD(\alpha)$ and $\mathcal{A}_H(\sigma) = \max\{\mathcal{A}_H(\bar{x}) \,|\, \sigma_{\bar{x}} \neq 0\}$ for $\sigma = \sum \sigma_{\bar{x}}\bar{x} \in CF_*(H)$. Just like critical value selectors, one can formulate a geometrical interpretation of the action selectors, where they take the various capped one-periodic orbits representing a particular cohomology class and push the "energy" down as far as possible.

From the above definitions we point out some of their useful properties.

- Projective invariance: $c^{\lambda\alpha}(H) = c^\alpha(H)$ for any $\lambda \in \mathbb{Q}$, $\lambda \neq 0$.
- Symplectic invariance: $c^\alpha(\phi^* H) = c^\alpha(H)$ for any symplectic diffeomorphism ϕ.
- Lipschitz continuous: c^α is Lispschitz continuous in the C^0-topology on the space of Hamiltonians H. In particular, $|c^\alpha(H) - c^\alpha(K)| \leq \|H - K\|$, where $\|\cdot\|$ is the Hofer norm.
- Triangle inequality: $c^{\alpha*\beta}(H\#K) \leq c^\alpha(H) + c^\beta(K)$.
- Hamiltonian shift: $c^\alpha(H + a(t)) = c^\alpha(H) + \int_0^1 a(t)dt$, where $a : S^1 \to \mathbb{R}$.
- Homotopy invariance: Let H and K be two Hamiltonians which are homotopic to each other, then we have $c^\alpha(H) = c^\alpha(K)$, for all $\alpha \in QH^*(H)$.
- Spectrality: When M is a rational manifold and H is a one-periodic Hamiltonian on M, then $c^\alpha(H) \in \mathcal{S}(H)$.

Let $\widetilde{\mathcal{H}am}(M,\omega)$ be the universal covering space for the group of Hamiltonian diffeomorphisms $\mathcal{H}am(M,\omega)$. It is worth mentioning that one can also look at the action selectors c^α as functions from $\widetilde{\mathcal{H}am}(M,\omega)$ to the reals ([Oh05]).

Remark 2.3. We also point out that one can define the action selectors on the homology of M for any Hamiltonian H. In the non-degenerate case one can define the action selector on the elements $u \in HQ_*(M)$ by $c_u(H) = \inf_{[\sigma]=u} \mathcal{A}_H(\sigma)$ for $\sigma = \sum a_{\bar{x}}\bar{x} \in CF_*(M)$. The action selectors defined on the homology also satisfy similar properties to the ones on cohomology. The details for the homology case are outlined in [GG09] and [Oh05]. There is one property in particular which interests us: $c_u(H) = c^{LS}_u(H)$ for $u \in H_*(M)$ and for H an autonomous and C^2-small Hamiltonian. Also, based on the definitions for action selectors on cohomology and homology we see they share the relationship $c^\alpha(H) = c_{PD(\alpha)}(H)$. Putting these two facts together we end up with $c^\alpha(H) = c^{LS}_{PD(\alpha)}(H)$ when H is autonomous and C^2-small.

2.5. **Alexander-Spanier cohomology.**

Our last preliminary that needs to be introduced is a version of cohomology due to J.M. Alexander and E.H. Spanier. We will be primarily following the exposition given in [HZ11], [Mas91], [Spa81].

Begin by fixing a subspace $A \subset M$ and define $\mathcal{O}_A$ to be the set of all open neighborhoods of the subset A. One is then able to define an ordered structure on this set in the following manner: for $U, V \in \mathcal{O}_A$ we say $U \leq V$ if and only if $V \subseteq U$. We call $(\mathcal{O}_A, \leq)$ the *directed system of neighborhoods for the set* A.

Now let $\mathcal{C}$ to be the category of all subspaces of the manifold M and the category $\mathcal{A}$ to be an algebraic category, which, for our purposes, will either be the category of abelian groups, the category of commutative rings, or the category of modules over a fixed ring. Define a continuous functor $H\colon \mathcal{C} \to \mathcal{A}$ that takes continuous maps $f\colon V \to U$, for $U, V \in \mathcal{C}$ and maps it to a homomorphism $H(f)\colon H(U) \to H(V)$. If $U \leq V$ we can define the inclusion map $i_{VU}\colon H(U) \to H(V)$. From any directed system $\mathcal{O}_A$ we define $D_A := \bigoplus_{U \in \mathcal{O}_A} H(U)$ and the homomorphism $j_U\colon H(U) \to D_A$ as the inclusion map into the U-th component of D_A. Next take K_A to be the subring that is generated by elements of the form $j_U(\alpha_U) - j_V i_{VU}(\alpha_U)$ for $U \leq V$, $\alpha_U \in H(U)$. We denote the quotient of D_A by K_A by $dir\ \lim_{U \in \mathcal{O}_A} H(U) := D_A/K_A$, which we call the *direct limit of* A.

We define $\bar{H}^*(A;\mathbb{Z}) := dir\ \lim_{U \in \mathcal{O}_A} H^*(U;\mathbb{Z})$ to be the *Alexander-Spanier cohomology* for the subspace $A \subseteq M$, where $H^*(U;\mathbb{Z})$ is the usual singular cohomology. The restriction maps from $H^k(U;\mathbb{Z})$ to $H^k(A;\mathbb{Z})$ end up defining a natural homomorphism from $\bar{H}^k(A;\mathbb{Z})$ to $H^k(A;\mathbb{Z})$. When this homomorphism is an isomorphism that holds for all k and any coefficient group, then we say the subspace A is *taut in* M. The following result gives us a useful list of criteria for when A will be taut in the manifold M.

Theorem 2.4. *In each of the following four cases the subspace A is taut in M:*

- *A is compact and M is Hausdorff.*
- *A is closed and M is paracompact Hausdorff.*
- *A is arbitrary and every open subset of M is paracompact Hausdorff.*
- *A is a retract of some open subset of M.*

3. Proofs of Theorems 1.1 and 1.3

We are now in a position to present the proofs for Theorems 1.1 and 1.3. We will begin by showing the monotone case result and then present the aspherical one.

Proof of Theorem 1.3.

We start by looking at the fixed points of ϕ_H which have associated action equal to $c^\alpha(H) = a$ and call this set F_a. Let $\delta > 0$ be small and define $F_{(a-\delta, a+\delta)}$ to be the set of all fixed points of ϕ_H that have their associated action in the interval $(a-\delta, a+\delta)$. We then take U_δ to be a neighborhood of the set $F_{(a-\delta,a+\delta)}$. We want to show $H^k(U_\delta) \neq 0$ for some $1 \leq k \leq 2n$ and for δ close to 0.

Suppose not and that $H^k(U_\delta) = 0$ for all $0 < k \leq 2n$ in order to arrive at a contradiction. Let $h\colon M \to \mathbb{R}$ be a C^2-small function on M where h is identically equal to zero on the neighborhood U_δ and outside of this set it is strictly negative. Let $\eta \in H^k(M)$, where $deg(\eta) > 0$. We can approximate the function h by a sequence of Morse functions that are at least C^2-small, call them h_n, such that $h_n \to h$ as $n \to \infty$ in the C^0-topology and for a fixed $x \in F_a$ we have $h_n(x) = 0$ only at this single point and strictly negative everywhere else. By making use of the fact that $c^{LS}_{PD(\eta)}(h_n) < 0$ for all $\eta \in H^k(M)$, with $k > 0$, and since c^η is Lipschitz in the C^0-topology we have $c^\eta(h) < -\delta_h < 0$ for all $\eta \in H^k(M)$ with $k > 0$ and δ_h is a positive constant depending on the function h. It is worth noting that we cannot say the same thing about $PD(\alpha)$ because it is possible that $PD(\alpha) = [M]$, which implies $c^\alpha(h) = c^{LS}_{[M]}(h) = max(h) = 0$.

Define $r\colon S^1 \to \mathbb{R}$ to be a nonnegative, C^2-small function, equal to zero outside of a small neighborhood of zero in S^1. Set $f_t = r(t)h$. This means that the Hamiltonian flow of f will be a reparametrization of the flow of h through time $\epsilon = \int_0^1 r(t)dt$.

Next we look at the family of Hamiltonians $H\#(sf)$ for $s \in [0,1]$. By the construction of f we have $H\#(sf) = H$ on the set U_δ, but outside of the set U_δ it is possible, for values of s close to 1, that $H\#(sf)$ has a 1-periodic orbit, say $\bar{x}$, such that $c^\alpha(H\#(sf)) = \mathcal{A}_{H\#(sf)}(\overline{x}) \neq a$. However, we claim that for small values of s that we can prevent this situation from occurring. In particular, we claim that one can find a nonzero s' in $[0,1]$ such that for all $0 \leq s \leq s'$ the Hamiltonians $H\#(sf)$ may have new 1-periodic orbits such that their action is not in $\mathcal{S}(H)$ and that their values may drift into the interval $(a-\delta, a+\delta)$, but by picking s' small enough these new critical values for $H\#(s'f)$ cannot drift into the neighborhood $(a-\frac{\delta}{2}, a+\frac{\delta}{2})$. We will show this fact below in Lemma 3.1 and suppose for the time being that such an s' exists. Then for all $0 \leq s \leq s'$ we have $\mathcal{S}(H\#(sf)) \cap (a-\frac{\delta}{2}, a+\frac{\delta}{2}) = \mathcal{S}(H) \cap (a-\frac{\delta}{2}, a+\frac{\delta}{2})$.

Now, when h and r are sufficiently C^2- small, ϵh and f have the same periodic orbits, which are the critical points of h, and they have the same action spectrum. This is also true for the functions $\epsilon s'h$ and $s'f$. The same will be true for every function in the linear family $\tilde{f}_l = (1-l)\epsilon s'h + ls'f$, with $l \in [0,1]$, connecting $\epsilon s'h$ and $s'f$. Using the continuity property of c_u, the fact that each $\mathcal{S}(\tilde{f}_l)$ is a set of measure zero, and that $\mathcal{S}(\tilde{f}_1) = \mathcal{S}(\tilde{f}_l)$ for all l, we conclude that $c_{PD(\beta)}(s'f) = c_{PD(\beta)}(\epsilon s'h) < 0$.

We again use the continuity property of c^α and that the sets $\mathcal{S}(H\#(sf))$ have measure zero for all s to give us $c^\alpha(H\#(s'f)) = c^\alpha(H) = a$. Since $c^{\alpha*\beta}(H) = c^\alpha(H)$ we have $c^{\alpha*\beta}(H\#(s'f)) = c^{\alpha*\beta}(H)$ as well. We then use the following triangle inequality for action selectors to give

$$c^{\alpha*\beta}(H) = c^{\alpha*\beta}(H\#(s'f)) \leq c^\alpha(H) + c^\beta(s'f) = c^\alpha(H) + c_{PD(\beta)}(s'f) < c^\alpha(H)$$

and creates a contradiction to the fact that $c^{\alpha*\beta}(H) = c^\alpha(H)$. This means we must have $H^k(U_\delta) \neq 0$ for $k = deg(\beta)$.

Now let $\mathcal{O}_{F_a}$ be a directed system of neighborhoods for the set F_a. Then Theorem 2.4 along with the basic properties of Alexander-Spanier cohomology and direct limits are outlined in Section 2.5 almost immediately implies $H^k(F) \neq 0$ for $k = deg(\beta)$, which proves Theorem 1.3. □

With the above in mind, we are able to prove Theorem 1.1.

Proof of Theorem 1.1.

First recall the assumption that M is symplectically aspherical, let $CL(M) = m$ and let $\alpha_1, \cdots, \alpha_m$ be cuplength representative in $H^{*>0}(M)$. Using a result from [GG09] which in the symplectically aspherical case says that for $\alpha, \beta \in H^*(M)$ with $deg(\beta) > 0$ we have $c^{\alpha\cup\beta}(H) \leq c^\alpha(H)$. This gives us the following monotonically decreasing sequence

$$c^{\tilde{\alpha}_m}(H) \leq \cdots \leq c^{\tilde{\alpha}_1}(H) \leq c^{\tilde{\alpha}}(H)$$

with $\tilde{\alpha} = PD([M])$, $\tilde{\alpha}_1 = \alpha_1$, $\tilde{\alpha}_2 = \alpha_2 \cup \tilde{\alpha}_1$, $\cdots$, $\tilde{\alpha}_m = \alpha_m \cup \tilde{\alpha}_{m-1}$. Since each $c^\beta(H) \in \mathcal{S}(H)$ and $\#\mathcal{S}(H) \leq m$ it implies there must be equality somewhere in the above chain of inequalities. So, $c^{\tilde{\alpha}_{i+1}}(H) = c^{\tilde{\alpha}_i}(H)$ for some $1 \leq i \leq m$, or

$\tilde{\alpha}_i = PD([M])$. Since $\tilde{\alpha}_{i+1} = \alpha_{i+1} \cup \tilde{\alpha}_i$ we just rename $\alpha_{i+1} = \beta$ and $\tilde{\alpha}_i = \alpha$ for notational convenience. This means $c^{\alpha \cup \beta}(H) = c^{\alpha}(H)$ and as we have pointed out in Section 2.3 the quantum product in the symplectically aspherical case reduces to the cup product, i.e. $\alpha * \beta = \alpha \cup \beta$, so we can apply Theorem 1.3, which immediately gives us our result. $\square$

Lemma 3.1. *There exists some nonzero s' in $[0,1]$ such that $\mathcal{S}(H\#(sf))$ does not gain any new critical values within the interval $(a - \frac{\delta}{2}, a + \frac{\delta}{2})$ for all $0 \leq s \leq s'$.*

Proof. Suppose not and we cannot find such a number s'. This means we can find a sequence of s_n in $[0,1]$ where $s_n \to 0$ as $n \to \infty$ and that there exists a one-periodic orbit x_n of $X_{H\#(s_n f)}$ such that $\mathcal{A}_{H\#(s_n)f}(\overline{x}_n) = a_n$ with $lim_{n\to\infty} a_n = a$. Now, since $H\#(s_n f) = H$ on the set U_δ, it means the fixed points for $\phi_{H\#(s_n f)}$, with associated action $a_n \in \mathcal{S}(H\#(s_n f))$, can't be elements of the set U_δ.

Our next step is to show we can find a one-periodic orbit x_* for X_H with $\mathcal{A}(\overline{x}_*) = a$ that comes from some subsequence of the x_n's. In order to show this we will use the generalized Arzela Ascoli theorem for metric spaces which says the following: If X_1 is compact Hausdorff space, X_2 is a metric space, $C(X_1, X_2)$ be the set of continuous functions from X_1 to X_2, and let $\{f_n\}$ be a sequence of functions in $C(X_1, X_2)$ that is uniformly bounded and equicontinuous, then there exists a subsequence $\{f_{n_j}\}$ that converges uniformly. We apply this to our capped loops $\bar{x}_n$, taking $X_1 = [0,1]$ and $X_2 = M$. Let d be the distance function that comes from the Riemannian metric g on M. We want to first show that there exists some real number $L > 0$ such that $d(x_n(t), x_n(s)) \leq L|t-s|$ for all n. Note that since the manifold M is compact that there is a uniform bound on the $X_{H\#(s_n f)}$ where $\|X_{H\#(s_n f)}\| \leq L$ for some $L > 0$ and for all n. Since the distance between two points $p, q \in M$ is given by $d(p,q) = \inf_\gamma(L(\gamma))$ for $L(\gamma) = \int_a^b \|\dot{\gamma}(t)\| dt$ we have $d(x_n(t), x_n(s)) \leq \int_s^t \|\dot{x}_n(u)\| du = \int_s^t \|X_{H\#(s_n f)}(x_n)\| du \leq L|t-s|$. This shows that the family of curves $\{x_n\}$ is uniformly Lipschitz, which implies that this family of curves is uniformly bounded and equicontinuous. This means there is a subsequence $\{x_{n_j}\}$ that converges to the curve x_*. The curve x_* is only a continuous loop from $[0,1]$ to M, but we can use the following result which tells us that x_* is actually a smooth solution to X_H.

Proposition 3.2. *Assume that the sequence of Hamiltonian vector fields $X_{H_n} \to X_H$ as $n \to \infty$ in the C^0-topology and x_n is a solution to X_{H_n} and $x_n \to x_*$ in the C^0-topology. Then x_* is a solution to X_H.*

This means x_* is a one-periodic solution to X_H. Our next step is to show that $\mathcal{A}(H)(\overline{x}_*) = a$. Let $\epsilon > 0$. Since $\mathcal{A}_{H\#(s_n f)}(\overline{x}_n) = a_n$ we can find some N_1 such that for all $n > N_1$ we get $|a_n - a| < \frac{\epsilon}{3}$. At the same time, the Hamiltonians $H\#(s_n f) \to H$ in the C^1-topology and we can find some N_2 such that for all $n > N_2$ we have $|\mathcal{A}_H(\overline{x}) - \mathcal{A}_{H\#(s_n f)}(\overline{x})| < \frac{\epsilon}{3}$. Lastly, since the x_{n_j} converge uniformly to the one-periodic solution x_* of X_H we can find some N_3 such that for all $n_j > N_3$ we get that $|\mathcal{A}_H(\overline{x}_*) - \mathcal{A}_H(\overline{x}_{n_j})| < \frac{\epsilon}{3}$. Then for $N = max\{N_1, N_2, N_3\}$ we have for $n > N$ that $|\mathcal{A}_H(\overline{x}_*) - a| \leq |\mathcal{A}_H(\overline{x}_*) - \mathcal{A}_H(\overline{x}_{n_j})| + |\mathcal{A}_H(\overline{x}_{n_j}) - \mathcal{A}_{H\#(s_{n_j} f)}(\overline{x}_{n_j})| + |\mathcal{A}_{H\#(s_{n_j} f)}(\overline{x}_{n_j}) - a| < \epsilon$. Since this is true for every $\epsilon > 0$ it gives $\mathcal{A}_H(\overline{x}_*) = a$.

Our next step is to show that the fixed point $x_*(0) = x_*(1)$ for ϕ_H that has associated action $\mathcal{A}_H(\overline{x}_*) = a$ is a point that is outside of the U_δ. In order to do this we will look at the other fixed points p_{n_j} for $\phi_{H\#(s_{n_j} f)}$ that come from the

loops x_{n_j}. In order to simplify the notation we will just relabel the points p_{n_j} to be p_n. Now, since M is a compact metric space we know that it is sequentially compact, meaning any sequence $\{y_n\}$ has a convergent subsequence $\{y_{n_j}\}$, and that the collection of points $\{p_n\}$ has a convergent subsequence $\{p_{n_j}\}$ that converges to the point p. In fact, the limit point p is a fixed point for ϕ_H, which we will show. Let $\epsilon > 0$ and we show that $d(\phi_H(p), p) < \epsilon$. Since $s_n f \to 0$ pointwise as $n \to \infty$ and since ϕ_H is continuous it implies that $\phi_{H\#(s_{n_j} f)} = \phi_H \circ \phi_{s_{n_j} f} \to \phi_H$ pointwise as $j \to \infty$. Then there exists some N_1 such that for all $n_j > N_1$ we have $d(\phi_H(p), \phi_{H\#(s_{n_j} f)}(p)) < \frac{\epsilon}{3}$. We can also find some N_2 such that for all $n_j > N_2$ that $d(\phi_{H\#(s_{n_j} f)}(p), \phi_{H\#(s_{n_j} f)}(p_{n_j})) < \frac{\epsilon}{3}$ and we can find an N_3 such that for all $n_j > N_3$ we get $d(p_{n_j}, p) < \frac{\epsilon}{3}$. For $n_j > N = max\{N_1, N_2, N_3\}$ we end up with $d(\phi_H(p), p) \leq d(\phi_H(p), \phi_{H\#(s_{n_j} f)}(p)) + d(\phi_{H\#(s_{n_j} f)}(p), \phi_{H\#(s_{n_j} f)}(p_{n_j})) + d(p_{n_j}, p) < \epsilon$. So, p is a fixed point for ϕ_H.

Let x be the loop formed by the curve $\phi^t_H(p)$ for $0 \leq t \leq 1$. Since both x_* and x are one-periodic solutions for X_H and they both have the point p on them, then by uniqueness of solutions of O.D.E.'s it forces $x_* = x$. Then, we end up with p being a fixed point of ϕ_H, which is on the curve x, and has the associated action $\mathcal{A}_H(\overline{x}) = a$. However, we have that p is the limit point of the points p_{n_j} and we know that $p_{n_j} \notin U_\delta$ for all n and means that $p \notin U_\delta$, which creates a contradiction. □

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